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TRANSIENT THERMAL RESPONSE OF TWO SOLIDS IN CONTACT OVER A CIRCULAR DISK

SATWINDAR SINGH SADHAL

Department of Mechanical Engineering and Applied Mechanics, University of Pennsylvania, 220 South 33rd Street, Philadelphia, PA 19104, U.S.A.

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NOMENCLATURE

- a, radius of disk of contact;
- Fo_i, $\kappa_i t/a^2$ = Fourier number;
- k_1, thermal conductivity of solid 1;
- k_2, thermal conductivity of solid 2;
- p, Laplace transform parameter;
- $P_n(x)$, Legendre polynomial of degree n;
- r, cylindrical radial coordinate;
- R, thermal resistance;
- R_{ss}, steady-state thermal resistance;
- t, time;
- T_1, temperature distribution in solid 1;
- T_2, temperature distribution in solid 2;
- T_{10}, initial temperature in solid 1;
- T_{20}, initial temperature in solid 2;
- z, cylindrical coordinate.

Greek symbols

- $\delta_{j,k}$, Kronecker delta;
- ε, oblate spheroidal coordinate;
- η, oblate spheroidal coordinate;
- θ_1, dimensionless temperature distribution in solid 1;
- θ_2, dimensionless temperature distribution in solid 2.

1. INTRODUCTION

IT IS WELL known that when the plane surfaces of two bodies are brought together the actual contact does not take place over the entire interfacial area, but over a fraction of that area. As a result, for any heat flow taking place across the interface, the flow lines are constricted and lead to what is termed 'contact resistance'. Over the region where there is no contact, heat may be transported by the air present between the surfaces. At atmospheric pressures, however, this heat transfer may be neglected, and it is quite reasonable to assume that the areas of no contact are insulated. Under such an assumption we have a well-defined, heat-conduction problem but the general irregularity of the contacting areas makes an exact analysis virtually impracticable. For cases in which the actual contact takes place over a small fraction of the total interfacial area, one can model a single area of contact by assuming it to be a circular disk between two semi-infinite solids. The areas around the disk can be taken to be insulated.

The steady-state solution for heat flow in this geometry is well known [1] and the transient case has been dealt with numerically by Schneider *et al.* [2]. In an approximate analysis, Heasley [3] obtained a solution to the transient case by assuming the region of contact to be a perfectly conducting sphere between two semi-infinite solids. Other models by Heasley [3] involve one-dimensional approximations of the heat equation and their validity is very restricted. In the present study the problem as posed by Schneider *et al.* [2] is solved analytically by using a long-time perturbation scheme based on the work done by Norminton and Blackwell [4] for heat flow from an isothermal disk. That is, a solution to the time-dependent heat equation is found for the problem in which two different solids at different uniform initial temperatures are brought into contact over a finite circular disk. The solution is valid for long time, large thermal diffusivities or for small areas of contact; in other words it is valid for large Fourier numbers.

2. STATEMENT OF PROBLEM

Two semi-infinite solids at different initial temperatures are brought together and perfect thermal contact is established over a finite circular region. The rest of the areas of the contacting planes are assumed to be insulated. Far away from the contact areas the temperature in each solid is taken to be fixed at the initial value.

To adapt to the geometry one resorts to the oblate spheroidal coordinate system. The prolate and oblate coordinate systems were used by Norminton and Blackwell [4] to obtain the large-time temperature distribution for one-medium heat flow from isothermal spheroids and the isothermal circular disk. In the present analysis the case of the disk is generalized to that of two media, with the disk temperature being nonuniform and time-varying.

With the transformation of the (r, z) cylindrical coordinates into the (ε, η) oblate spheroidal coordinates through the relations $r = a[(1+\varepsilon^2)(1-\eta^2)]^{1/2}$ and $z = a\varepsilon\eta$, the axisymmetric heat equation takes the form:

$$\frac{1}{a^2(\varepsilon^2+\eta^2)}\left\{\frac{\partial}{\partial\varepsilon}\left[(1+\varepsilon^2)\frac{\partial T_i}{\partial\varepsilon}\right]+\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial T_i}{\partial\eta}\right]\right\}=\frac{1}{\kappa_i}\frac{\partial T_i}{\partial t}, \tag{1}$$

$$t>0;\ 0<\varepsilon<\infty;\ 0<\eta\le 1,$$

where i is used to denote $i = 1, 2$; t is the time, T_1 is the temperature distribution in the hotter solid (say, solid 1), T_2 is the temperature distribution in the other solid (solid 2), and k_1 and k_2 are the corresponding thermal diffusivities. The initial, the boundary and the interface conditions are given by:

$$\left.\begin{array}{l}T_1 = T_{10}\\ T_2 = T_{20}\end{array}\right\},\quad t=0;\ 0\le\varepsilon<\infty;\ 0\le\eta\le 1, \tag{2}$$

$$\left.\begin{array}{c}T_1 = T_2\\ k_1\dfrac{\partial T_1}{\partial\varepsilon} = -k_2\dfrac{\partial T_2}{\partial\varepsilon}\end{array}\right\},\quad \varepsilon=0;\ t>0;\ 0\le\eta\le 1, \tag{3}$$

$$\left.\begin{array}{l}T_1 = T_{10}\\ T_1 = T_{20}\end{array}\right\},\quad \varepsilon\to\infty;\ t>0;\ 0\le\eta\le 1, \tag{4}$$

and

$$\frac{\partial T_1}{\partial \eta} = \frac{\partial T_2}{\partial \eta} = 0, \quad \eta = 0; \ t > 0; \ 0 \le \varepsilon < \infty. \tag{5}$$

Here, k_1 and k_2 are the thermal conductivities of the solids 1 and 2, respectively.

3. ANALYSIS

After redefining the dependent variables T_1 and T_2 in the dimensionless form

$$\theta_i = \frac{T_i - T_{20}}{T_{10} - T_{20}} - \delta_{1,i} \tag{6}$$

we take the Laplace transform of θ_i with respect to time, and then take the Legendre transform with respect to η. The resulting dependent variable

$$\Phi_{i,2m}(\varepsilon) = \int_0^1 P_{2m}(\eta) \int_0^\infty e^{-pt}\theta_i(\varepsilon, \eta; t)\mathrm{d}t\,\mathrm{d}\eta \tag{7}$$

is further substituted by $f_{i,2m}(\varepsilon) = \Phi_{i,2m}(\varepsilon)\exp[(p/\kappa_i)^{1/2} a\varepsilon]$ and the solution is obtained by regular perturbation expansion for $f_{i,2m}(\varepsilon)$ in powers of $p^{1/2}$.

Since the procedure for obtaining the perturbation solution is rather cumbersome (see Sadhal [5]), the details are left out for brevity. After carrying out the expansion to order $p^{3/2}$, and inversion to the $(\varepsilon, \eta; t)$ domain, we expand the solution for $a\varepsilon/(4\kappa_i t)^{1/2} \ll 1$ to give

$$\begin{aligned}\theta_i(\varepsilon, \eta, t) = {} & A_i \cot^{-1}\varepsilon + \left\{\frac{B_i\kappa_i^{1/2}}{a}\cot^{-1}\varepsilon - A_i\right\}\frac{a}{(\pi\kappa_i t)^{1/2}} \\ & + \left[A_i\left[\frac{1}{6}\varepsilon^3\cot^{-1}\varepsilon + \frac{1}{12}\varepsilon^2 - \frac{7}{18}\right]\right. \\ & - \frac{B_i\kappa_i^{1/2}}{a}\left[\left(\frac{1}{12}\varepsilon^2 - \frac{1}{3}\right)\cot^{-1}\varepsilon + \frac{1}{6}\varepsilon\right] \\ & + \frac{D_i\kappa_i}{2a^2} + \left\{\frac{A_i}{48}[(\varepsilon^3 - \varepsilon)\cot^{-1}\varepsilon - \varepsilon^2]\right. \\ & \left. + \frac{5F_i\kappa_i}{8a^2}[3\varepsilon^2 + \varepsilon)\cot^{-1}\varepsilon - 3\varepsilon^2]\right\} \\ & \left.\times(3\eta^2 - 1) - \frac{H_i\kappa_i^{1/2}}{2a^3}\cot^{-1}\varepsilon\right]\frac{a^3}{(\pi\kappa_i^3 t^3)^{1/2}} + \ldots,\end{aligned} \tag{8}$$

where

$$A_i = \frac{2(2i-3)}{\pi}\left[1 - \frac{k_i}{k_1 + k_2}\right], \tag{9}$$

$$B_i = -aA_i\left[\frac{A_1}{\kappa_1^{1/2}} - \frac{A_2}{\kappa_2^{1/2}}\right], \tag{10}$$

$$\begin{aligned}(k_1 + k_2)D_i = {} & a^2\left[\frac{2(k_1 + k_2)}{3\kappa_i}A_i + \frac{k_1 A_1}{3\kappa_1} + \frac{k_2 A_2}{3\kappa_2}\right] \\ & + (3 - 2i)\frac{2}{\pi}ak_2\left[\frac{B_1}{\kappa_1^{1/2}} - \frac{B_2}{\kappa_2^{1/2}}\right],\end{aligned} \tag{11}$$

$$F_i = -\frac{\pi a^2}{60}A_i\left[\frac{A_1}{\kappa_1} - \frac{A_2}{\kappa_2}\right] \tag{12}$$

and

$$\begin{aligned}(k_1 + k_2)H_i = {} & a^2\left[\frac{k_1 B_1}{\kappa_1} + \frac{k_2 B_2}{\kappa_2}\right] + (3 - 2i)\left\{(k_1 + k_2 - k_i)\right. \\ & \times\frac{2}{3}a^2\left[\frac{B_1}{\kappa_1} - \frac{B_2}{\kappa_2}\right] + a\left[\frac{D_1}{\kappa_1^{1/2}} - \frac{D_2}{\kappa_2^{1/2}}\right] \\ & \left. - \frac{7}{9}a^3\left[\frac{A_1}{\kappa_1^{3/2}} - \frac{A_2}{\kappa_2^{3/2}}\right]\right\}.\end{aligned} \tag{13}$$

For the special case $k_1 \to \infty$ we have

$$\theta_1 = 0, \tag{14}$$

and:

$$\begin{aligned}\theta_2 = {} & \frac{2}{\pi}\left[\cot^{-1}\varepsilon + \frac{1}{\varepsilon}\left[\frac{2}{\pi}\cot^{-1}\varepsilon - 1\right]\frac{a\varepsilon}{(\pi\kappa_2 t)^{1/2}}\right. \\ & + \left\{\frac{1}{12}\cot^{-1}\varepsilon + \frac{1}{6\varepsilon} - \frac{1}{6\pi\varepsilon}\cot^{-1}\varepsilon - \frac{1}{3\pi\varepsilon^2} + \frac{5}{36\varepsilon^3}\right. \\ & + \frac{2}{\pi^2\varepsilon^3} + \frac{1}{\varepsilon^3}\left[\frac{7}{9\pi} + \frac{4}{\pi^3}\right]\cot^{-1}\varepsilon \\ & \left.\left. + \frac{1}{4\varepsilon}[\varepsilon\cot^{-1}\varepsilon - 1]\eta^2\right\}\frac{a^3\varepsilon^3}{(\pi\kappa_2^3 t^3)^{1/2}}\right] + \ldots\end{aligned} \tag{15}$$

This result corresponds to the situation in which a semi-infinite solid, initially at a uniform temperature T_{20}, has a circular region at the surface exposed to a temperature T_{10} for $t > 0$; the remainder of the surface is insulated. The results of Norminton and Blackwell [4] agree with equation (15) only up to order $a\varepsilon/(\pi\kappa_2 t)^{1/2}$. For the next higher order terms, the error in [4] is due to the omission of a $p^{3/2}\exp[-(p/\kappa^2)^{1/2} a\varepsilon]$ type term in the perturbation expansion. This term actually makes a contribution of the order $a^3\varepsilon^3/(\pi\kappa_2 t)^{3/2}$. The details are discussed by Sadhal [5].

4. CALCULATION OF RESISTANCE

After heat flow calculations over the disk, the resistance of the two solids is found to be

$$\begin{aligned}R = R_{ss}\Bigg\{ & 1 + \left(\frac{2}{\pi}\right)\frac{1}{[1 + (k_2/k_1)]}\left[\frac{k_2}{k_2}\left(\frac{\kappa_2}{\kappa_1}\right)^{1/2} + 1\right](\pi Fo_2)^{-1/2} \\ & + \frac{1}{[1 + (k_2/k_1)]^2}\left[\frac{4}{9}\left\{\left(\frac{\kappa_2}{\kappa_1}\right)^{3/2} + \left(\frac{k_2}{k_1}\right)^2\right\}\right. \\ & - \frac{2}{9}\left(\frac{k_2}{k_1}\right)\left\{\left(\frac{\kappa_2}{\kappa_1}\right)^{3/2} + 1\right\} - \left(\frac{2}{\pi}\right)^2\frac{1}{[2 + (k_2/k_1)]} \\ & \left.\times\left\{\left(\frac{k_2}{k_1}\right)\left(\frac{\kappa_2}{\kappa_1}\right)^{1/2} + 1\right\}^3\right](\pi Fo_2)^{-3/2} + \ldots\Bigg\}^{-1},\end{aligned} \tag{16}$$

where $R_{ss} = (k_1 + k_2)/4a(T_{10} - T_{20})k_1 k_2$ is the steady-state resistance and $Fo_2 = \kappa_2 t/a^2$ is the Fourier number based on the lower thermal diffusivity (i.e. $\kappa_2 < \kappa_1$). This result is presented in Fig. 1 for contacts between copper and steel, steel and glass, and copper and glass, with Fo_2 ranging between 1 and 10000. The calculations show that the resistances for copper–glass and steel–glass contacts are almost the same. This behavior results because, for these cases, most of the resistance is that of the glass alone and the metals (steel or copper) make very little contribution.

5. DISCUSSION

The present work represents a generalization and a correction of the results obtained by Norminton and Blackwell[4]. In a numerical analysis by Schneider *et al.* [2], four combinations of solids were dealt with. In each case the ratio of the time-dependent resistance to the steady-state resistance was plotted as a function of the Fourier number based on the lower thermal diffusivity. They showed that when the resistances ratio is plotted against the variable,

$$X = [1 + (\kappa_2/\kappa_1)^{1/2}][\kappa_1\kappa_2/(\kappa_1 + \kappa_2)]t/\alpha^2,$$

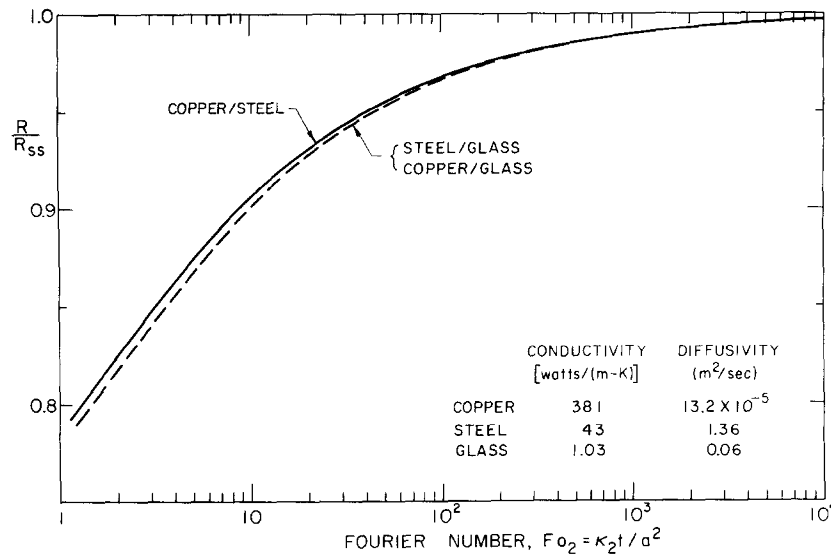

FIG. 1. Unsteady thermal resistances of different pairs of solids as a function of the Fourier number.

all the four cases lie close to a single curve. This curve, however, is not consistent with the individual plots they obtained, but it agrees approximately with equation (16) of the present study for Fourier numbers greater than unity.

The thermal resistance, in general, depends on the conductivity ratio, k_1/k_2, the diffusivity ratio, κ_1/κ_2, and the Fourier number $\kappa_2 t/a^2$, where we take $\kappa_2 < \kappa_1$. Schneider *et al.* [2] showed that for the cases they considered, the influence of the conductivity ratio appeared only in the steady-state part of the thermal resistance. They concluded that when the resistance is normalized with the steady-state value, the result depends only on the diffusivity ratio and the Fourier number. In the present study, however, equation (16) shows that the conclusion of Schneider *et al.* [2] is not necessarily valid. The results of Schneider *et al.* [2] showing no dependence on k_2/k_1 is not surprising, because they considered materials which, like most solids, have $k_2/k_1 \simeq \kappa_2/\kappa_1$. The present study, however, is completely general and it exposes the dependence on both k_2/k_1 and κ_2/κ_1.

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ON THE TRANSITION TO TRANSVERSE ROLLS IN INCLINED INFINITE FLUID LAYERS — STEADY SOLUTIONS

DOUGLAS W. RUTH

Department of Mechanical Engineering, University of Calgary, Calgary, Alberta T2N 1N4, Canada

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NOMENCLATURE

Gr, Grashof number;
H, spacing between plates;
Pr, Prandtl number;
x, y, z, spatial coordinates;
T, temperature;
T_0, reference temperature.

Greek

α, spatial wavenumber;